# 附录Ⅱ　重要平面曲线

(1) 三次抛物线　(2) 半立方抛物线

(3) 概率曲线　(4) 箕舌线　(5) 蔓叶线

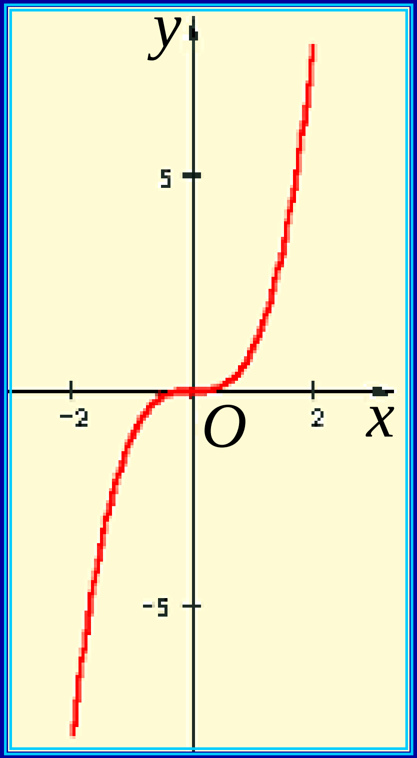
(6) 笛卡儿叶形线　(7) 星形线　(8) 摆线

(9) 心形线　(10) 双曲螺线　(11) 对数螺线

(12) 阿基米德螺线　(13) 伯努利双纽线

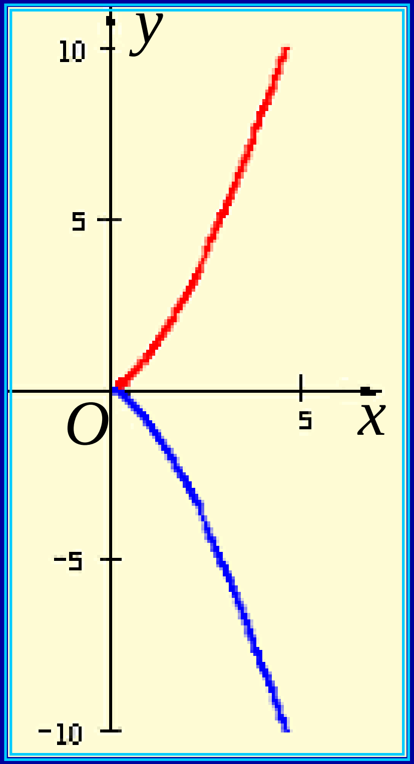
(14) 三叶玫瑰线　(15) 四叶玫瑰线

三次抛物线 $y = x^3$




- 拐点: (0, 0)
- 关于原点对称

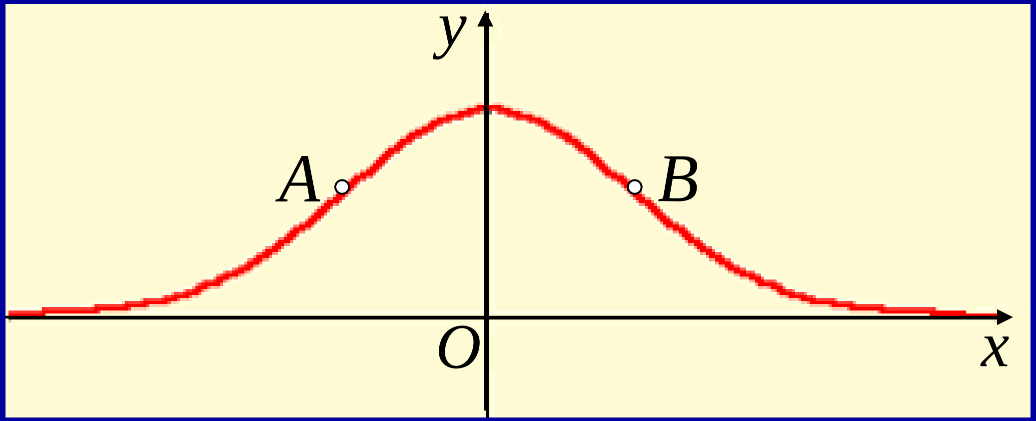
半立方抛物线 $y^2 = x^3$




- 尖点: (0, 0)
- 在尖点处与 $x$ 轴相切
- 关于 $x$ 轴对称

# 概率曲线

$$y = \mathrm{e}^{-x^2}$$




- 拐点: $(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{\mathrm{e}}})$
- 拐点处切线斜率: $\mp\sqrt{\frac{2}{\mathrm{e}}}$
- 渐近线: $x$ 轴
- 与 $x$ 轴之间的面积: $\sqrt{\pi}$
- 关于 $y$ 轴对称

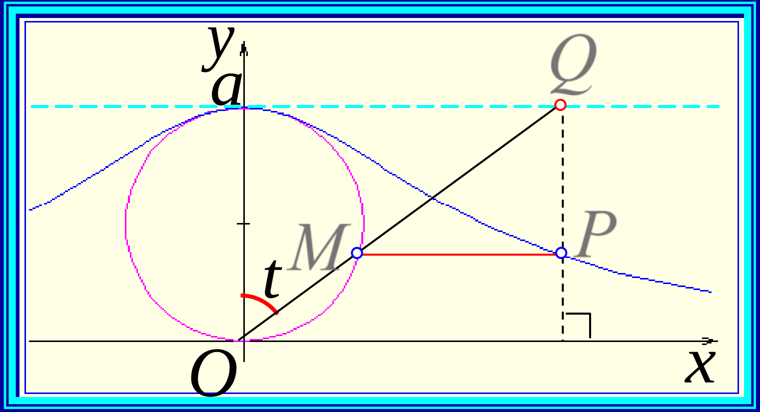
设 $\xi$ 服从标准正态分布，则其概率密度函数为

$$f(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$$

- 拐点: $(\pm 1, \frac{1}{\sqrt{\mathrm{e}}})$
- 与 $x$ 轴之间的面积: 1

**箕舌线** $y=\dfrac{a^3}{x^2+a^2}$

或 $\begin{cases} x = a\tan t \\ y = a\cos^2 t \end{cases}$



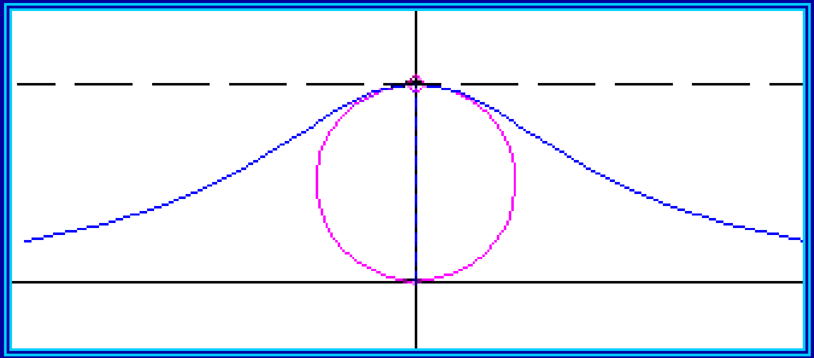

- 轨迹：

$M$是直径为$a$ 的圆上的动点,

$Q$是射线$OM$与 $y = a$ 的交点,

$QP\perp x$ 轴， $MP\parallel x$ 轴

$P$点轨迹即为箕舌线 .



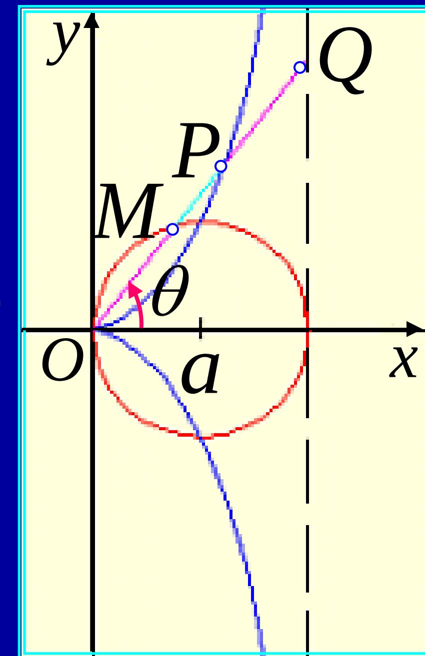
点击图中任意点
动画开始或暂停

- 渐近线: $y = 0$
- 曲线与渐近线之间的面积:

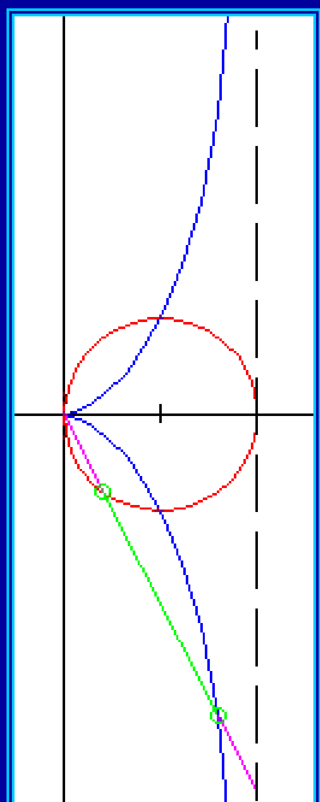
$$S=\pi\, a^2$$

## 蔓叶线 $y^2(2a-x)=x^3$

或 $\begin{cases} x = a\dfrac{at^2}{1+t^2} \\ y = a\dfrac{at^3}{1+t^2} \end{cases}$ $(t=\tan\theta)$




点击图片任意处播放开始或暂停



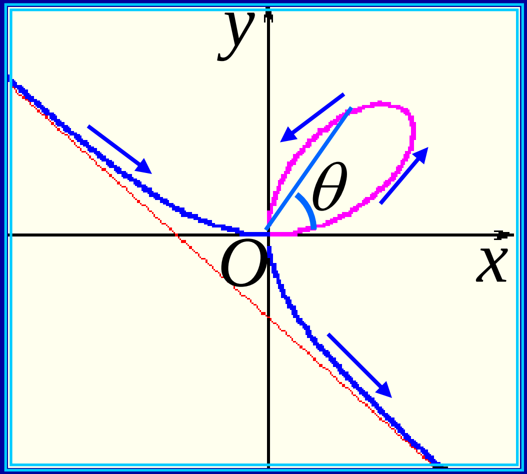
- 轨迹:

  $M$ 是半径为 $a$ 的母圆上的动点，满足 $OM = PQ$ 之点 $P$ 的轨迹即为蔓叶线

- 渐近线: $x = 2a$
- 曲线与渐近线之间的面积: $S = 3\pi\, a^2$

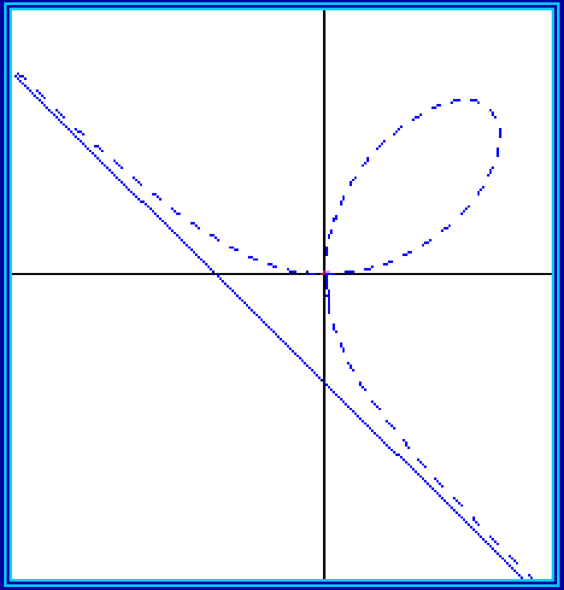
# 笛卡儿叶形线

$$\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases} \qquad t \neq -1$$




动画开始或暂停

点击图中任意点



动画走向: $-\infty \to -1$

$-1 \to +\infty$

参数的几何意义: $t = \tan\theta$

$t \in (-\infty, -1) \to \theta \in (-\frac{\pi}{2}, -\frac{\pi}{4})$

图形在第四象限

$t \in (-1, 0] \to \theta \in (\frac{3\pi}{4}, \pi]$

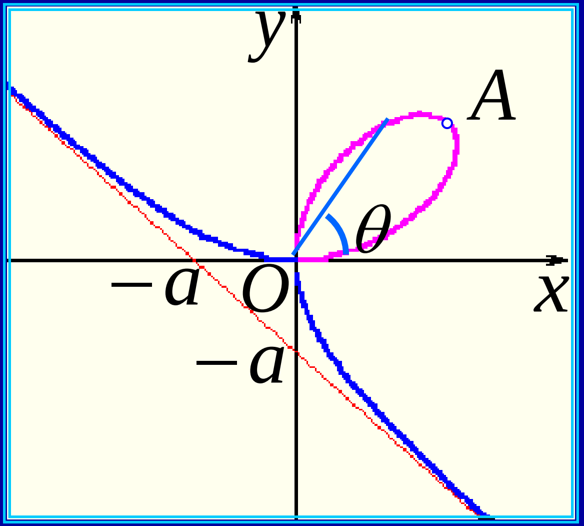
图形在第二象限

$t \in [0, +\infty) \to \theta \in [0, \frac{\pi}{2})$

图形在第一象限

# 笛卡儿叶形线(续)

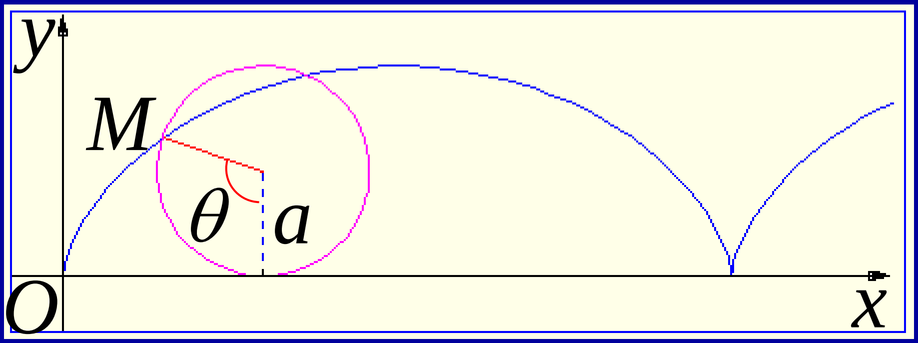
$$\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases} \qquad t \neq -1$$




- 结点: $O(0,0)$

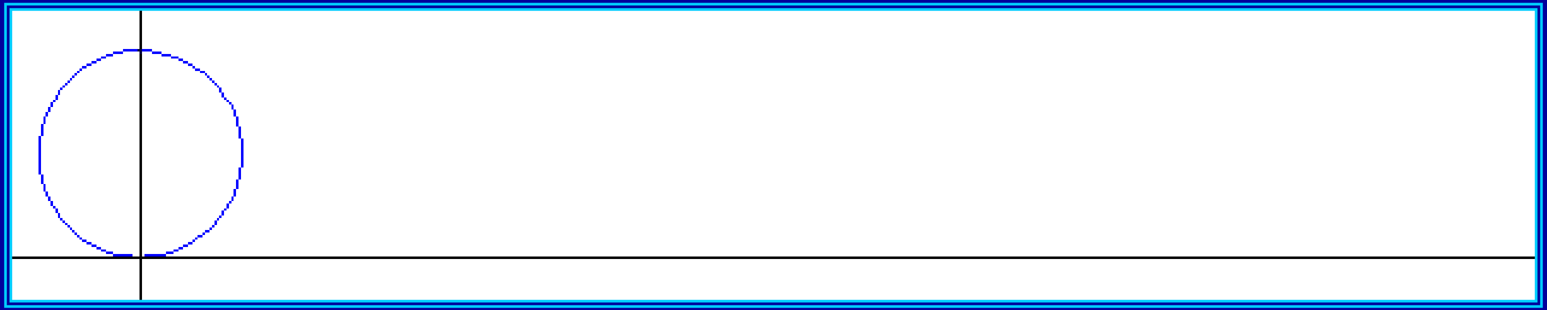
  在该点与 $x$ 轴 $y$ 轴相切，曲率半径为 $\frac{3}{2}a$
- 顶点: $A(\frac{3a}{2},\frac{3a}{2})$
- 渐近线: $x+y+a=0$
- 圈套所围面积: $S_1=\frac{3}{2}a^2$
- 曲线与渐近线之间的面积: $S_2=\frac{3}{2}a^2$

**摆线** $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$



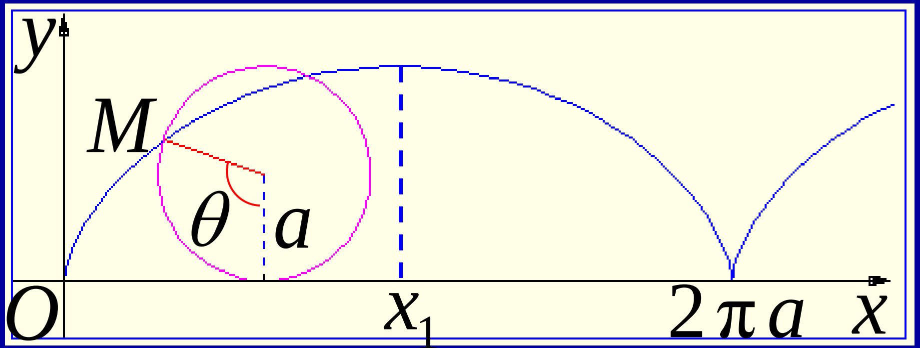

- 轨迹:

  半径为 $a$ 的圆周沿直线无滑动地滚动时，其上定点 $M$ 的轨迹即为摆线 .



点击图中任意点动画开始或暂停

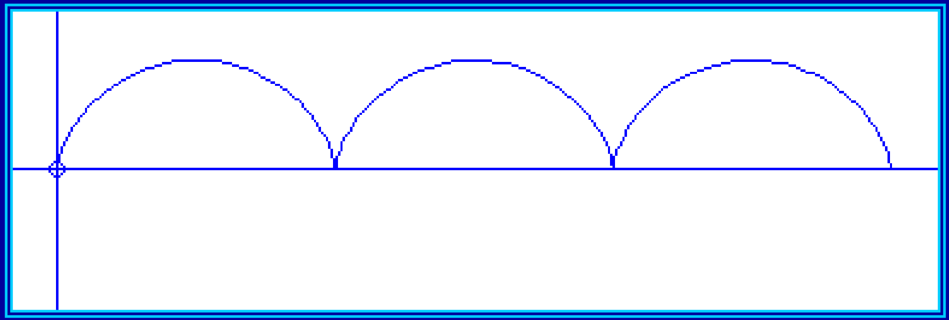
**摆线(续)** $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$

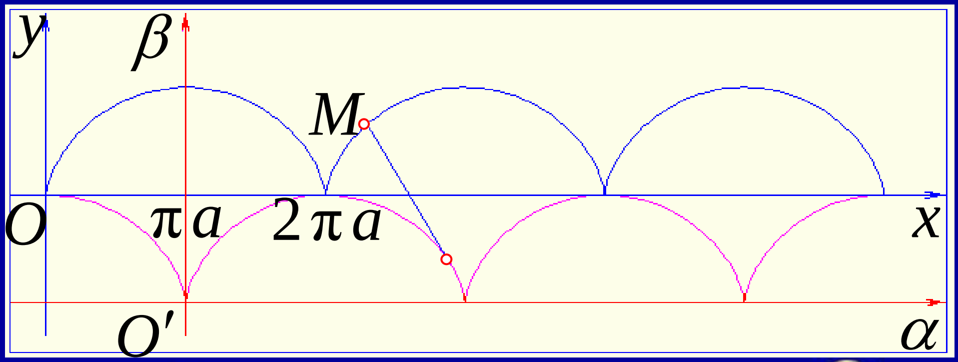



- 周　期: $T = 2\pi a$
- 极大点: $x_k = (2k-1)\pi a \quad (k = 1, 2, \cdots)$
- 曲率半径: $R = 4a\sin\frac{t}{2}$
- 一拱长: $8a$
- 一拱面积: $S = 3\pi a^2$
- 渐屈线: 仍为摆线,

  在$\alpha O'\beta$坐标系下

  与原摆线一致





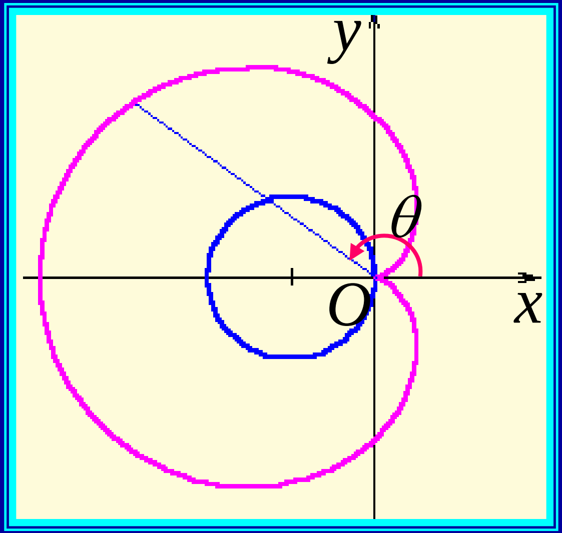


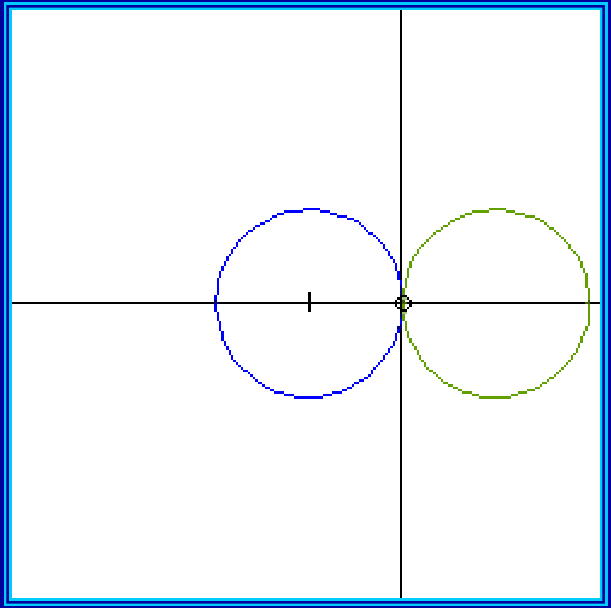

结束

## 心形线

$$x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$$

或 $\quad r = a(1-\cos\theta)$



点击图中任意点
动画开始或暂停

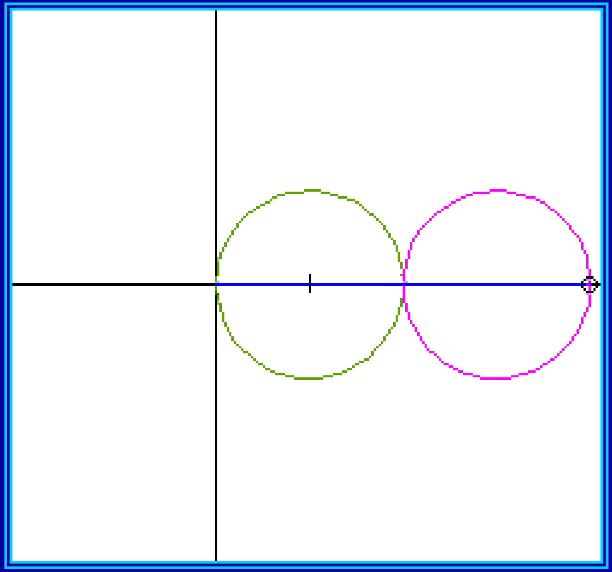
- 轨迹: 外摆线的一种

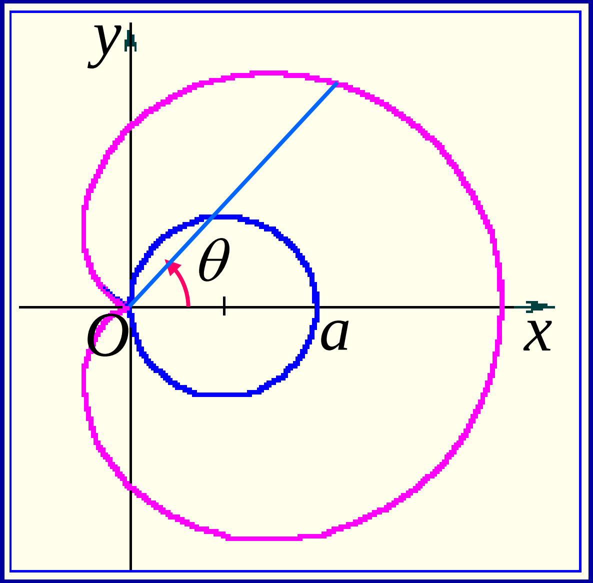
  动圆直径 = 定圆直径 = $a$
- 尖点: (0, 0)
- 面积: $\frac{3}{2}\pi a^2$
- 弧长: $8a$

# 心形线的另一种形式

$$x^2 + y^2 - ax = a\sqrt{x^2 + y^2}$$

即 $r = a(1+\cos\theta)$





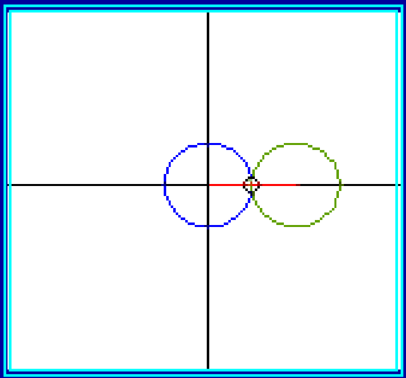

点击图中任意点
动画开始或暂停

- 尖点: (0, 0)
- 面积: $\frac{3}{2}\pi a^2$
- 弧长: $8a$

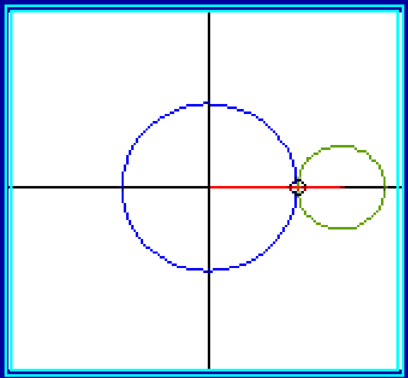
## 外摆线 (圆外旋轮线) 族

$$\begin{cases} x = (a+b)\cos t - b\cos\frac{a+b}{b}t \\ y = (a+b)\sin t - b\sin\frac{a+b}{b}t \end{cases}$$

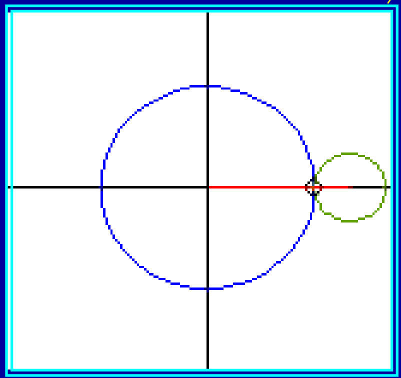
定圆圆心为 (0,0), 半径为 $a$, 动圆半径为 $b$, $m = b/a$



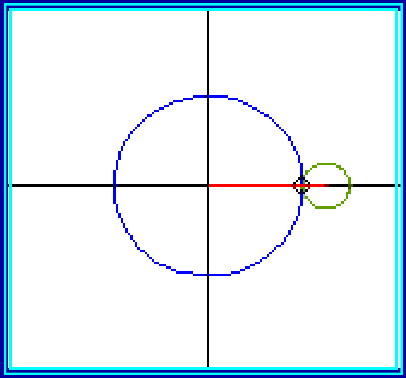
$m = 1$为心形线



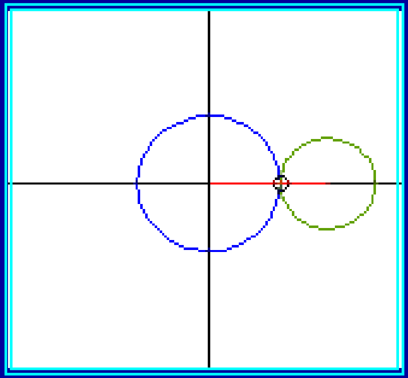
$m = 2$



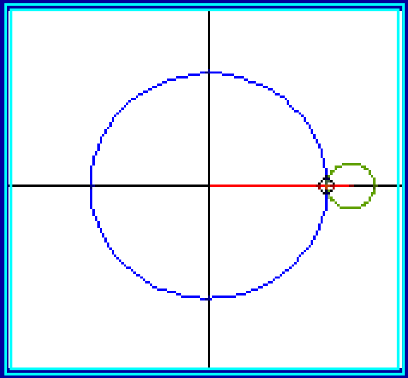
$m = 3$



$m = 4$



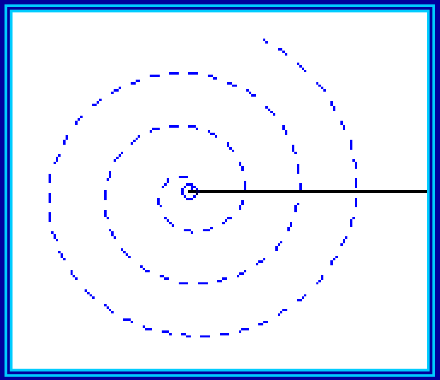
$m = \frac{3}{2}$



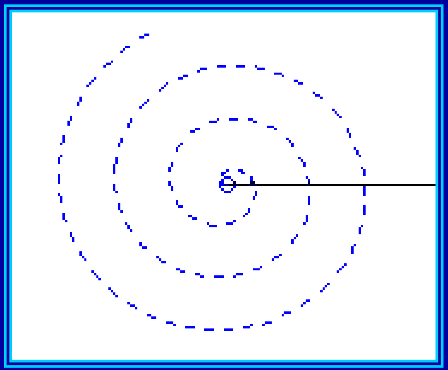
$m = 5$

点击图中任意点动画开始或暂停

## 阿基米德螺线 $r = a\theta$



$a > 0$

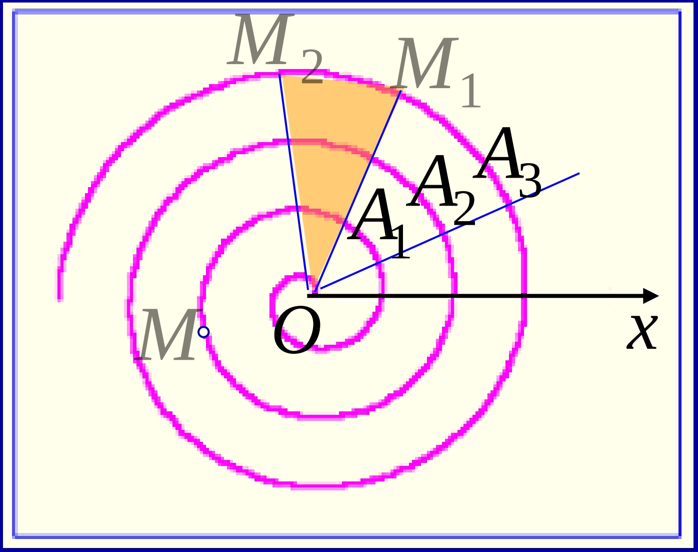


$a < 0$

- 物理意义：动点 $M$ 以常速 $v$ 沿一射线运动，该射线又以定速 $\omega$ 绕极点转动时，点$M$ 的轨迹即为阿基米德螺线

$$r = \frac{v}{\omega}\theta$$

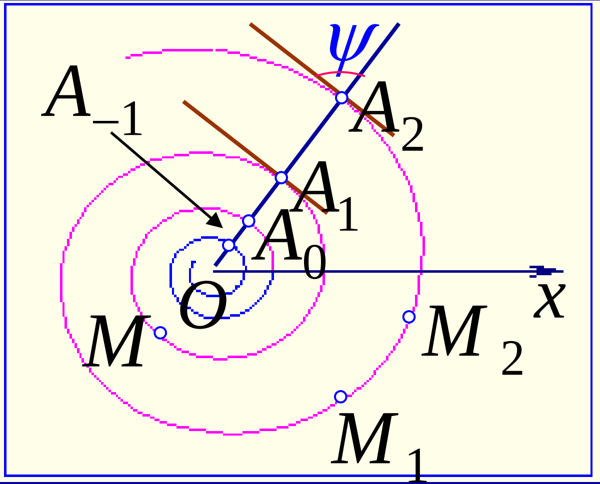
# 阿基米德螺线(续)




- 等距性：过极点的射线与曲线交于 $A_1, A_2, A_3, \cdots,$ 它们之间的间隔都是 $2\pi a$
- 弧　长：$L_{OM} = \frac{a}{2}(\theta\sqrt{\theta^2+1} + \operatorname{arsh}\theta)$

  其中 $\operatorname{arsh}\theta = \ln(\theta + \sqrt{1+\theta^2})$
- 曲率半径：$R = a\dfrac{(\theta^2+1)^{\frac{3}{2}}}{\theta^2+2}$
- 扇形 $M_1OM_2$ 的面积：$S = \frac{1}{6}a^2(\theta_1^2 - \theta_2^2)$

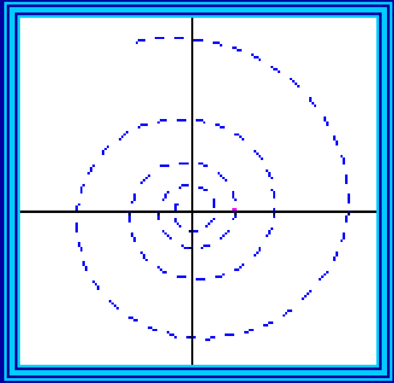
## 对数螺线(等角螺线) $r = e^{a\theta}$




- 等角性：曲线与所有过极点的射线的交角 $\psi$ 都相等: $\tan\psi = \frac{1}{a}$
- 等比性: 过极点的射线与曲线交于 $\cdots, A_{-1}, A_0, A_1, \cdots$ 则 $\cdots, OA_{-1}, OA_0, OA_1, \cdots$ 各线段成等比级数, 公比为 $e^{2a\pi}$
- 弧长：$L_{M_1M_2} = \frac{\sqrt{1+a^2}}{a}(r_2 - r_1)$

$$L_{OM} = \frac{\sqrt{1+a^2}}{a} r$$

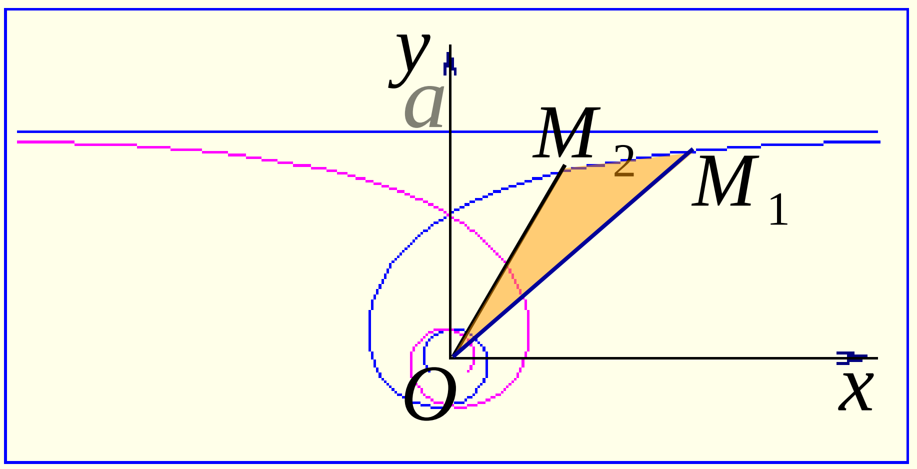
- 曲率半径：$R = \sqrt{1+a^2}\, r$



点击图中任意点动画开始或暂停

动画走向为 $\theta : 0 \to +\infty$

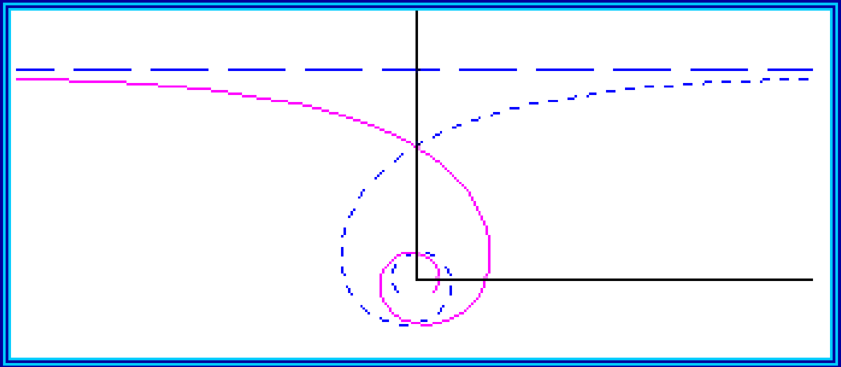
## 双曲螺线 $r = a/\theta$




- 曲线由两支组成，它们关于 $y$ 轴对称
- 渐近点：极点 $O$ $(\theta \to \pm\infty)$
- 渐近线：$y = a$
- 曲率半径：$R = \frac{a}{\theta}\left(\frac{\sqrt{1+\theta^2}}{\theta}\right)^3$

动画开始或暂停

点击图中任意点



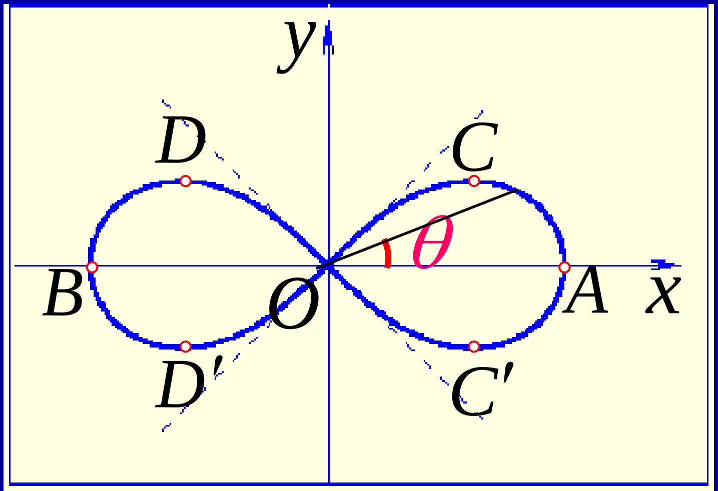
动画走向为 $\theta : 0^+ \to +\infty$

- 扇形 $M_1 O M_2$ 的面积：

$$S = \frac{a^2}{2}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$$

# 伯努利双纽线 $(x^2+y^2)^2 = a^2(x^2-y^2)$

或 $r^2 = a^2\cos 2\theta$




- 结点(同拐点) : $O(0,0)$

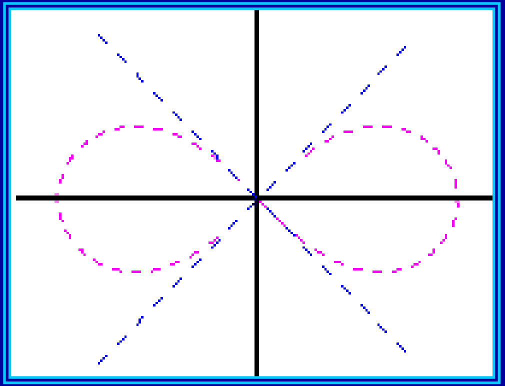
  在该点的切线斜率为$\pm 1$

- 顶　点: $A, B(\pm a, 0)$
- 极值点: $\pm\frac{\sqrt{6}}{4}a$

  极　值 : $\pm\frac{\sqrt{2}}{4}a$

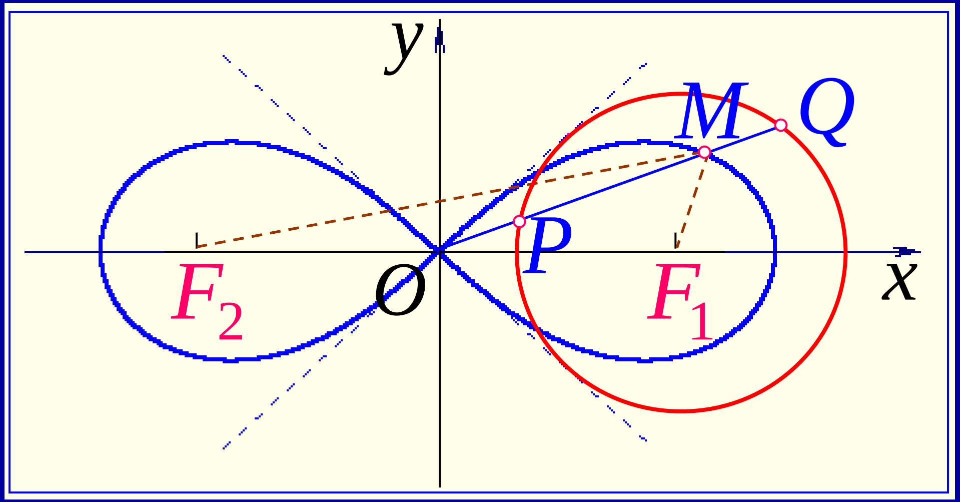
  对应点: $C, C', D, D'$

- 曲率半径: $\frac{a^2}{3r}$
- 双纽面积: $a^2$



点击图中任意点
动画开始或暂停

# 伯努利双纽线的轨迹特点




$$OF_1 = OF_2 = \frac{a}{\sqrt{2}}$$

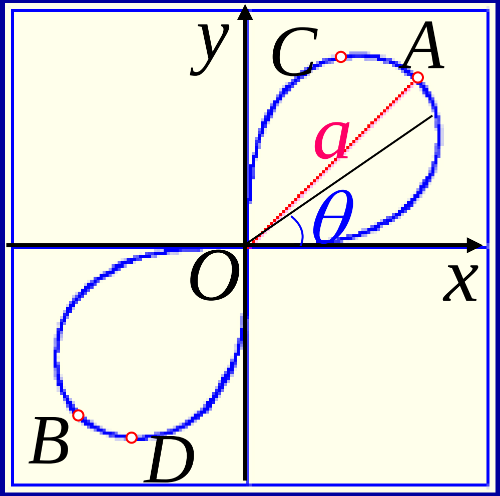
- 双纽线上的点 $M$ 满足： $MF_1 \cdot MF_2 = \frac{1}{2}a^2$
- 以 $F_1$ 为圆心， $\dfrac{1}{2}a$ 为半径作圆, 自 $O$ 作射线交圆于 $P, Q$

  则双纽线右支上的点满足： $OM = PQ$

  由对称性，左支也有类似结果

# 伯努利双纽线的另一形式 $(x^2+y^2)^2=2a^2xy$

即 $r^2=a^2\sin 2\theta$




- 结点(同拐点)：$O(0,0)$

  在该点的切线为 $x, y$ 轴

- 顶点: $A, B(\pm\frac{\sqrt{2}}{2}a, \pm\frac{\sqrt{2}}{2}a)$

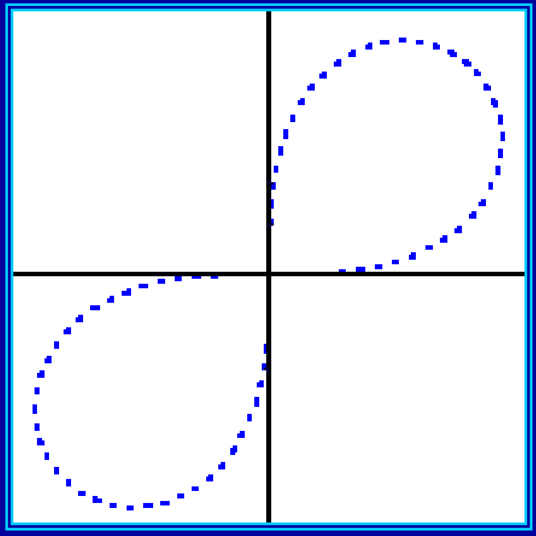
- 极值点: $x=\pm\frac{a}{4}\sqrt[4]{12}$

  极　值：$y=\pm\frac{a}{4}\sqrt[4]{108}$

  对应点: $C, D$

- 曲率半径: $\frac{a^2}{3r}$

- 双纽面积: $a^2$

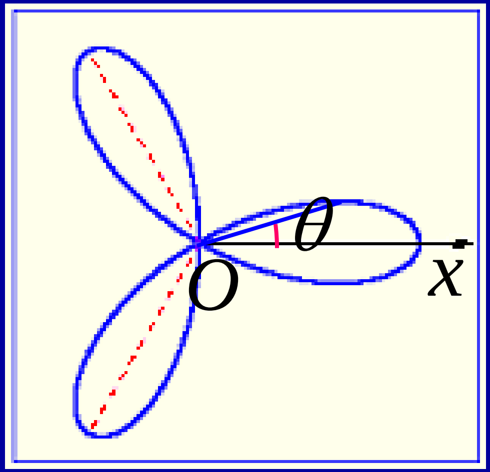


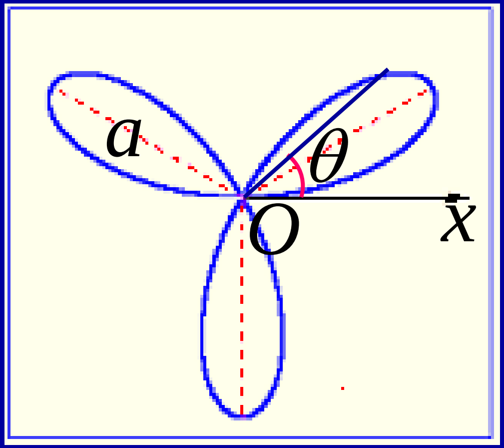
点击图中任意点动画开始或暂停

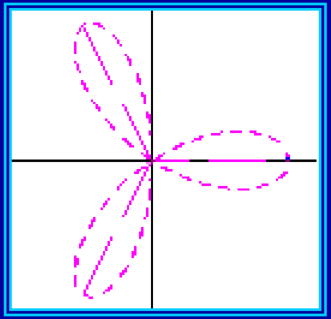
# 三叶玫瑰线

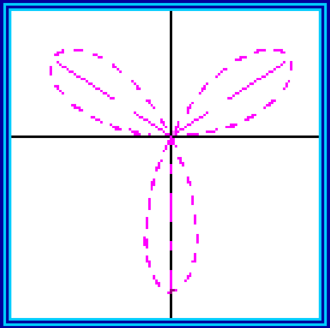
$$r = a\cos 3\theta,\ \theta \in [0, 2\pi]$$

$$r = a\sin 3\theta,\ \theta \in [0, 2\pi]$$





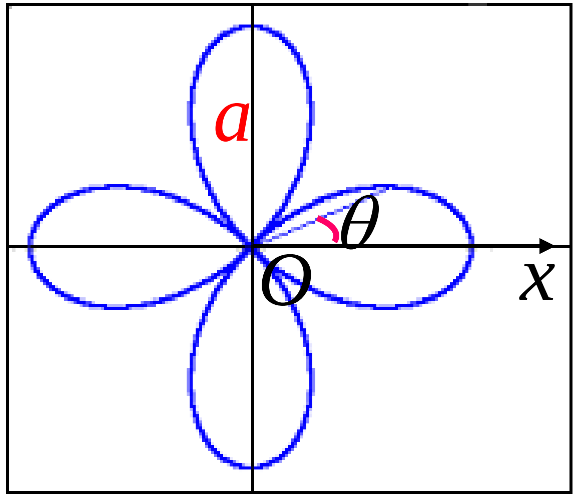
点击图中任意点
动画开始或暂停

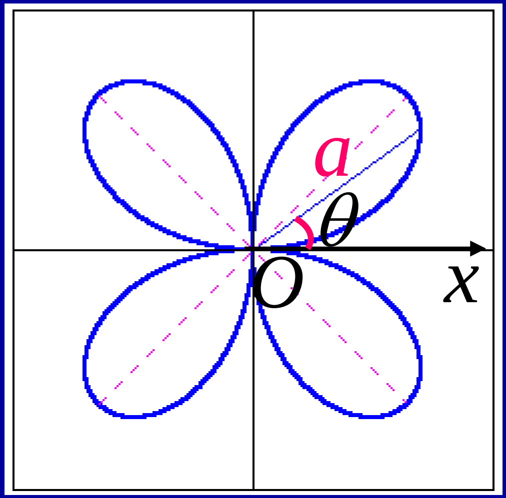


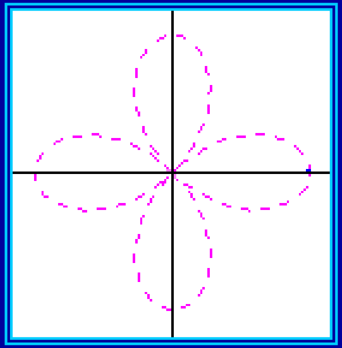
结束

# 四叶玫瑰线

$r = a\cos 4\theta, \theta \in [0, 2\pi]$ $\qquad$ $r = a\sin 4\theta, \theta \in [0, 2\pi]$



点击图中任意点
动画开始或暂停