专题

陈 现 平

E–mail: chxp123456789@163.com
blog:chxp123456789.blog.163.com

聊城大学　数学科学学院

November 11, 2009

目录

欧氏空间

Example 1.1

设α为欧氏空间V的一个非零向量,$\alpha_1, \cdots, \alpha_m \in V$,且

$$(\alpha_i, \alpha) > 0, (\alpha_i, \alpha_j) \leq 0, i, j = 1, 2, \cdots, i \neq j$$

证明:$\alpha_1, \cdots, \alpha_m$线性无关.

证: 设

$$k_1\alpha_1 + \cdots + k_{m\alpha_m} = 0.$$

不妨设$k_i \geq 0, 1 \leq i \leq r(\leq m), k_j \leq 0, r+1 \leq j \leq m$.令

$$\beta = k_1\alpha_1 + \cdots + k_r\alpha_r = -(k_{r+1}\alpha_{r+1} + \cdots + k_m\alpha_m)$$

则

$$\begin{aligned} 0 \leq (\beta, \beta) &= (k_1\alpha_1 + \cdots + k_r\alpha_r, -(k_{r+1}\alpha_{r+1} + \cdots + k_m\alpha_m)) \\ &= -\textstyle\sum_{i=1}^{r}\sum_{j=r+1}^{m} k_ik_j(\alpha_i, \alpha_i) \leq 0 \end{aligned}$$

从而

$$0 = \beta = k_1\alpha_1 + \cdots + k_r\alpha_r = -(k_{r+1}\alpha_{r+1} + \cdots + k_m\alpha_m)$$

于是

$$0=(\beta,\alpha)=(k_1\alpha_1+\cdots+k_r\alpha_r,\alpha)=\sum_{i=1}^{r}k_i(\alpha_i,\alpha)$$

$$0=(\beta,\alpha)=(k_{r+1}\alpha_{r+1}+\cdots+k_m\alpha_m,\alpha)=\sum_{j=r+1}^{m}k_j(\alpha_j,\alpha)$$

由条件可得$k_1=\cdots=k_m=0$.

Example 1.2

证明:n维欧氏空间中存在且至多存在$n+1$个向量两两夹角大于$90^{”}$.

证:

求证在n维欧式空间中两两夹钝角的向量的个数的最大值是$n+1$.

证明: 用归纳法分两步, 首先证明至多有$n+1$个向量两两夹钝角:

$n=1$是显然的, 设命题对$n-1$维欧式空间成立, 考察n维的情形: 如果有$n+2$个向量$\alpha_1, \alpha_2, \cdots, \alpha_{n+2}$两两之间夹钝角, 考虑$\alpha_1$及其生成的一维子空间$\{\alpha_1\}$的正交补$M^{\perp}$, 任何$\alpha_2, \cdots, \alpha_{n+2}$ 有唯一的表示

$$\alpha_i = c_i\alpha_1 + \beta_i \quad \beta_i \in M^{\perp}, i = 2, 3, \cdots, n+2$$

显然$c_i < 0$. 考察$n-1$维欧式空间$M^{\perp}$中的$n+1$个向量$\beta_2, \cdots, \beta_{n+2}$, 我们有

$$(\alpha_i, \alpha_j) = (c_i\alpha_1 + \beta_i, c_j\alpha_1 + \beta_j) = c_ic_j|\alpha_1|^2 + (\beta_i, \beta_j)$$

即

$$(\beta_i, \beta_j) = (\alpha_i, \alpha_j) - c_ic_j|\alpha_1|^2 < 0$$

即$\beta_2, \cdots, \beta_{n+2}$两两夹钝角, 但$M^{\perp}$的维数是$n-1$, 这与归纳假设矛盾! 所以$n$维欧式空间中至多有$n+1$个向量两两夹钝角.

其次证明至少有$n+1$个向量两两夹钝角: 仍是对n归纳, $n=1$显然, 设$n-1$维的时候结论成立, 考察n维的情形:

首先取一个$n-1$维的子空间M, 根据归纳假设, 在M内有n个两两夹钝角的向量$\alpha_1, \alpha_2, \cdots, \alpha_n$, 似乎只要再找一个向量与它们都夹钝角即可. 然而从$n=1$到$n=2$的情形告诉我们这走不通. (验证一下!). 所以要对$\alpha_1, \alpha_2, \cdots, \alpha_n$进行一下微调, 给它们同时加上一个向量. 设$\beta$是$M^{\perp}$中的非零单位向量,取正数$\lambda$使得其满足

$$-2\lambda^2 > \max\{(\alpha_i, \alpha_j), \quad 1 \leqslant i < j \leqslant n\}$$

那么

$$(\alpha_i - \lambda\beta, \beta) = -\lambda < 0$$

$$(\alpha_i - \lambda\beta, \alpha_j - \lambda\beta) = (\alpha_i, \alpha_j) + \lambda^2 < -\lambda^2 < 0$$

从而

$$\alpha_1 - \lambda\beta, \alpha_2 - \lambda\beta, \cdots, \alpha_n - \lambda\beta, \beta$$

就是满足要求的$n+1$个向量.

Example 1.3

(1)设$\alpha_1, \cdots, \alpha_m$为一组线性无关的列向量,经施密特正交化为$\beta_1, \cdots, \beta_m$.记$G(\alpha_1, \cdots, \alpha_m)$为$\alpha_1, \cdots, \alpha_m$的度量矩阵.则

$$|G(\alpha_1, \cdots, \alpha_m)| = |G(\beta_1, \cdots, \beta_m)| = ||\beta_1||^2 ||\beta_2||^2 \cdots ||\beta_m||^2$$

(2)设$A = (a_{ij})$为n阶实矩阵,求证

$$|A|^2 \leq \prod_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2$$

证: (1) 由正交化的过程可知

$$B = (\beta_1, \cdots, \beta_m) = (\alpha_1, \cdots, \alpha_m)C = AC$$

其中C为对角线元素为1的上三角矩阵.则注意到$\beta_1, \cdots, \beta_m$的正交性有

$$\begin{aligned} |G(\alpha_1, \cdots, \alpha_m)| &= |A^T A| \\ &= |C^T A^T AC| \\ &= |B^T B| \\ &= ||\beta_1||^2 ||\beta_2||^2 \cdots ||\beta_m||^2 \end{aligned}$$

(2)考虑$A^T A$,利用(1)的结论.

Example 1.4

设σ为n维欧氏空间V的对称变换.证明$V = Im\sigma \oplus \ker\sigma$.

Example 1.5

(武汉大学03)设f为n维欧氏空间V的对称变换,证明$\mathrm{Im}f = \ker f^{\perp}$

证: $\forall f(\alpha) \in Imf, \beta \in \ker f$,有

$$(f(\alpha), \beta) = (\alpha, f(\beta)) = 0$$

即$Imf \subseteq \ker f^{\perp}$,
又

$$\dim Imf = n - \dim\ker f = \dim(\ker f^{\perp})$$

故结论成立.

Example 1.6

设α,β为n维欧氏空间V的两个正交的列向量,且长度都是2.$A=E+\alpha\alpha^T+\beta\beta^T$.求$A$的特征多项式.

证:

3. **解** A的特征多项式为

$$
\begin{aligned}
f(\lambda) &= |\lambda E_n - A| \\
&= \left|(\lambda-1)E_n - \alpha\alpha^T - \beta\beta^T\right| \\
&= \left|(\lambda-1)E_n - \begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\alpha^T \quad \beta^T)\right| \\
&= (\lambda-1)^{n-2}\left|(\lambda-1)E_2 - \begin{pmatrix}\alpha^T \\ \beta^T\end{pmatrix}(\alpha \quad \beta)\right| \\
&= (\lambda-1)^{n-2}\left|(\lambda-1)E_2 - \begin{pmatrix}\alpha^T\alpha & \alpha^T\beta \\ \beta^T\alpha & \beta^T\beta\end{pmatrix}\right| \\
&= (\lambda-1)^{n-2}\begin{vmatrix}\lambda-1-\alpha^T\alpha & -\alpha^T\beta \\ \beta^T\alpha & \lambda-1-\beta^T\beta\end{vmatrix} \\
&= (\lambda-1)^{n-2}(\lambda^2-10\lambda+25+(\alpha^T\beta)^2).\ \blacksquare
\end{aligned}
$$

Example 1.7

设实数域上的n阶矩阵$A=(B,C)$,其中B为$n\times m$矩阵.证明:$|A|^2\leq |B^TB||C^TC|$.

Example 1.8

设W_1, W_2是n维欧氏空间V的子空间,$dimW_1 = n_1 < n_2 = dimW_2$.证明存在$0 \neq \alpha \in W_2, \alpha \perp W_1$.

证: 由于$W_2 + W_1^{\perp} \subset V$,故

$$dim(W_2 \cap W_1^{\perp}) = dimW_2 + dimW_1^{\perp} - dim(W_2 + W_1^{\perp}) \geq n_2 + n - n_1 - n > 0$$

从而存在$0 \neq \alpha \in W_2 \cap W_1^{\perp}$,故结论成立.

正交阵

Example 2.1

证明:不存在正交矩阵A, B,满足$A^2 = AB + B^2$.

证: 反证法.由$A^2 = AB + B^2$左乘A^T,右乘B^T可得

$$A - B = A^T B^2, A + B = A^2 B^T$$

从而$A + B, A - B$都是正交阵,故

$$(A + B)^T (A + B) = E, (A - B)^T (A - B) = E$$

展开后相加可得$4E = 2E$.矛盾.

Example 2.2

(北京理工04,太原科技06) A, B均为n阶正交矩阵,且$|A| = -|B|$,证明:

$$|A+B| = 0.(r(A+B)^* \le 1)$$

证: (法1)

$$\begin{aligned} |A||A+B| &= |A^T||A+B| = |E + A^T B| \\ &= |B^T B + A^T B| = |B^T + A^T||B| = |B||A+B| \end{aligned}$$

由于$|A| = -|B|$,以及$|B| \neq 0$,可得结论成立.
(法2)$|AB^T| = 1$,而AB^T为正交矩阵,又AB^T的实特征值为1或-1,虚特征值成对出现,故AB^T必有特征值-1, 从而

$$|-E - AB^T| = 0$$

即

$$|-B - A||B^T| = 0$$

从而结论成立.

Example 2.3

特征值都是实数的正交矩阵为对称矩阵.

Example 2.4

设A, B为n阶实正交矩阵,证明:$|A||B| = 1$的充要条件为$n - r(A + B)$为偶数.