同济大学

高等代数与解析几何

教师： 蒋志洪

理学院数学系

§7.2 线性变换的矩阵与相似矩阵

设V是数域$\mathbb{F}$ 上的线性空间，$(\alpha_1,\alpha_2,\cdots,\alpha_n)$是$V$ 一个有序基，T是V上的线性变换. 则T由$T(\alpha_1),T(\alpha_2),\cdots,T(\alpha_n)$唯一确定. 由于$(\alpha_1,\alpha_2,\cdots,\alpha_n)$是$V$ 一个有序基，所以$T(\alpha_1),T(\alpha_2),\cdots,T(\alpha_n)$ 可以由$\alpha_1,\alpha_2,\cdots,\alpha_n$唯一线性表示. 设

$$
\begin{aligned}
T(\alpha_1) &= a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n \\
T(\alpha_2) &= a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n \\
&\cdots\cdots\cdots\cdots \\
T(\alpha_n) &= a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n
\end{aligned}
$$

如果我们记$T(\alpha_1,\alpha_2,\cdots,\alpha_n) := (T(\alpha_1),T(\alpha_2),\cdots,T(\alpha_n))$, 则上面的等式可以写成矩阵形式

$$T(\alpha_1,\alpha_2,\cdots,\alpha_n) = (\alpha_1,\alpha_2,\cdots,\alpha_n)A$$

其中

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

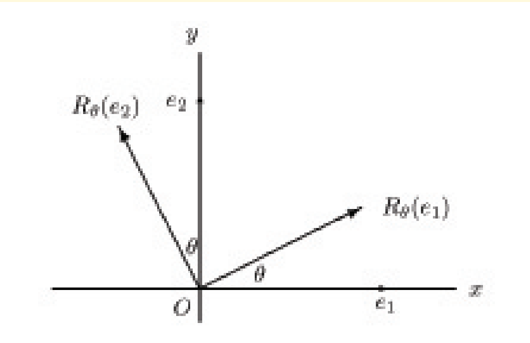
由于当有序基$(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 取定时，矩阵A由T唯一确定，所以以后我们将称矩阵A为线性变换T 在有序基$(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 下的矩阵.

例. 设R_θ 是$\mathbb{R}^2$ 上逆时针旋转θ 角的线性变换. 取定$\mathbb{R}^2$ 的一个有序基为

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

求R_θ 在有序基$(\mathbf{e}_1, \mathbf{e}_2)$ 下的矩阵.

解. 因为



$$R_\theta(\mathbf{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$$

$$R_\theta(\mathbf{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,$$

因此, R_θ 在有序基$(\mathbf{e}_1, \mathbf{e}_2)$ 下的矩阵为

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

即$R_\theta(\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{e}_1, \mathbf{e}_2)A$.

例. 设$M_2(\mathbb{F})$ 上线性变换$T(X)=X^t$. 求T 在有序基$(E_{11},E_{12},E_{21},E_{22})$ 下的矩阵(其中E_{ij} 是第i 行第j 列元素为1, 其余元素全为0 的2 阶矩阵).

解.

$$\begin{aligned}T(E_{11})&=1E_{11}+0E_{12}+0E_{21}+0E_{22}\\T(E_{12})&=0E_{11}+0E_{12}+1E_{21}+0E_{22}\\T(E_{21})&=0E_{11}+1E_{12}+0E_{21}+0E_{22}\\T(E_{22})&=0E_{11}+0E_{12}+0E_{21}+1E_{22}\end{aligned}$$

因此, T 在有序基$(E_{11},E_{12},E_{21},E_{22})$ 下的矩阵为

$$A=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

即$T(E_{11},E_{12},E_{21},E_{22})=(E_{11},E_{12},E_{21},E_{22})A$.

设V, W 分别是数域$\mathbb{F}$ 上的n 维与m 维线性空间, $B = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ 与$C = (\beta_1, \beta_2, \cdots, \beta_m)$ 分别是V 与W 的有序基, φ 是V 到W 的一个线性映射, 则B 中基向量在φ 下的象可以由C 这个基唯一地线性表示, 即

$$\varphi(\alpha_i) = \sum_{i=1}^{n} a_{ji}\beta_j, \quad i = 1, 2, \cdots, n,$$

写成矩阵形式, 有

$$\varphi(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\beta_1, \beta_2, \cdots, \beta_m) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

其中$m \times n$矩阵

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

称为线性映射φ 在有序基B 与C 下的矩阵.

练习. 习题7.2: 1(1)(只要做$\mathcal{A}_2$), 3, 6

引理. 设T是线性空间V上的线性变换, $\alpha_1, \alpha_2, \cdots, \alpha_m \in V$, $A = (a_{ij})$是$m \times s$矩阵, 则

$$T((\alpha_1, \alpha_2, \cdots, \alpha_m)A) = (T(\alpha_1, \alpha_2, \cdots, \alpha_m))A$$

证明. 对于$j = 1, 2, \cdots, s$, 设A_j为A的第j列列向量, 令

$$\beta_j = \sum_{i=1}^{m} a_{ij}\alpha_i = (\alpha_1, \alpha_2, \cdots, \alpha_m)A_j$$

它的矩阵形式

$$(\beta_1, \beta_2, \cdots, \beta_s) = (\alpha_1, \alpha_2, \cdots, \alpha_m)A$$

所以

$$\begin{aligned} T((\alpha_1, \alpha_2, \cdots, \alpha_m)A) &= T(\beta_1, \beta_2, \cdots, \beta_s) = (T(\beta_1), T(\beta_2), \cdots, T(\beta_s)) \\ &= (T(\sum_{i=1}^{m} a_{i1}\alpha_i), T(\sum_{i=1}^{m} a_{i2}\alpha_i), \cdots, T(\sum_{i=1}^{m} a_{is}\alpha_i)) \\ &= (\sum_{i=1}^{m} a_{i1}T(\alpha_i), \sum_{i=1}^{m} a_{i2}T(\alpha_i), \cdots, \sum_{i=1}^{m} a_{is}T(\alpha_i)) \\ &= (T(\alpha_1), T(\alpha_2), \cdots, T(\alpha_m))A = (T(\alpha_1, \alpha_2, \cdots, \alpha_m))A \end{aligned}$$

所以

$$T((\alpha_1, \alpha_2, \cdots, \alpha_m)A) = (T(\alpha_1, \alpha_2, \cdots, \alpha_m))A$$

定义. 设$(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 是数域$\mathbb{F}$ 上的线性空间V 的一个有序基, 定义集合$\mathcal{L}(V,V)$ 到集合$M_n(\mathbb{F})$ 的一个映射φ如下:

$$\varphi(T):=A;\quad T\in\mathcal{L}(V,V),A\text{是}T\text{在有序基}(\alpha_1,\alpha_2,\cdots,\alpha_n)\text{下的矩阵}$$

即

$$T(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T)$$

称φ为$\mathcal{L}(V,V)$的矩阵表示

定理. 设V 是数域$\mathbb{F}$ 上n 维线性空间, 映射φ 是$\mathcal{L}(V,V)$ 到$M_n(\mathbb{F})$ 的双射, 而且对于任意$T_1,T_2\in\mathcal{L}(V,V)$和$k\in\mathbb{F}$满足

(1) $\varphi(T_1+T_2)=\varphi(T_1)+\varphi(T_2)$

(2) $\varphi(kT_1)=k\varphi(T_1)$

(3) $\varphi(T_1T_2)=\varphi(T_1)\varphi(T_2)$

(4) $\varphi(\mathcal{E})=E$

(5) 当T_1是可逆线性变换时, $\varphi(T_1^{-1})=\varphi(T_1)^{-1}$.

说明. 定理的(1)和(2) 说明φ 是线性空间$\mathcal{L}(V,V)$ 到线性空间$M_n(\mathbb{F})$的同构.

证明. 设$\varphi(T_1)=\varphi(T_2)$则

$$\begin{aligned}(T_1(\alpha_1),T_1(\alpha_2),\cdots,T_1(\alpha_n))&=T_1(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1)\\(T_2(\alpha_1),T_2(\alpha_2),\cdots,T_2(\alpha_n))&=T_2(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_2)\end{aligned}$$

所以$T_1(\alpha_i)=T_2(\alpha_i),(i=1,2,\cdots,n)$, 由于线性变换由基上的取值唯一确定, 所以$T_1=T_2$, 即$\varphi$ 是单射.

设$A=(a_{ij})\in M_n(\mathbb{F})$, 对于$j=1,2,\cdots,n$, 令

$$\beta_j=\sum_{i=1}^{n}a_{ij}\alpha_i=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{nj}\end{pmatrix}$$

则由于线性变换由基上的取值唯一确定, 所以存在唯一的线性变换T 使得$T(\alpha_j)=\beta_j$, $(j=1,2,\cdots,n)$, 因此

$$\begin{aligned}T(\alpha_1,\alpha_2,\cdots,\alpha_n)&=(T(\alpha_1),T(\alpha_2),\cdots,T(\alpha_n))\\&=T(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A\end{aligned}$$

所以$\varphi(T)=A$, 即φ是满射.

(1)验证$\varphi(T_1+T_2)=\varphi(T_1)+\varphi(T_2)$

$$\begin{aligned}(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1+T_2)&=(T_1+T_2)(\alpha_1,\alpha_2,\cdots,\alpha_n)\\&=((T_1+T_2)(\alpha_1),(T_1+T_2)(\alpha_2),\cdots,(T_1+T_2)(\alpha_n))\\&=(T_1(\alpha_1)+T_2(\alpha_1),T_1(\alpha_2)+T_2(\alpha_2),\cdots,T_1(\alpha_n)+T_2(\alpha_n))\\&=(T_1(\alpha_1),T_1(\alpha_2),\cdots,T_1(\alpha_n))+(T_2(\alpha_1),T_2(\alpha_2),\cdots,T_2(\alpha_n))\\&=T_1(\alpha_1,\alpha_2,\cdots,\alpha_n)+T_2(\alpha_1,\alpha_2,\cdots,\alpha_n)\\&=(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1)+(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_2)\\&=(\alpha_1,\alpha_2,\cdots,\alpha_n)(\varphi(T_1)+\varphi(T_2))\end{aligned}$$

所以$\varphi(T_1+T_2)=\varphi(T_1)+\varphi(T_2)$

(2)验证$\varphi(kT_1)=k\varphi(T_1)$

$$\begin{aligned}(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(kT_1)&=(kT_1)(\alpha_1,\alpha_2,\cdots,\alpha_n)\\&=((kT_1)(\alpha_1),(kT_1)(\alpha_2),\cdots,(kT_1)(\alpha_n))\\&=(kT_1(\alpha_1),kT_1(\alpha_2),\cdots,kT_1(\alpha_n))\\&=k(T_1(\alpha_1),T_1(\alpha_2),\cdots,T_1(\alpha_n))\\&=k(T_1(\alpha_1,\alpha_2,\cdots,\alpha_n))\\&=k((\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1))\\&=(\alpha_1,\alpha_2,\cdots,\alpha_n)(k\varphi(T_1))\end{aligned}$$

所以$\varphi(kT_1)=k\varphi(T_1)$

(3)验证$\varphi(T_1T_2)=\varphi(T_1)\varphi(T_2)$

$$\begin{aligned}(\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1T_2) &= (T_1T_2)(\alpha_1,\alpha_2,\cdots,\alpha_n)\\ &= ((T_1T_2)(\alpha_1),(T_1T_2)(\alpha_2),\cdots,(T_1T_2)(\alpha_n))\\ &= (T_1(T_2(\alpha_1)),T_1(T_2(\alpha_2)),\cdots,T_1(T_2(\alpha_n)))\\ &= T_1(T_2(\alpha_1),T_2(\alpha_2),\cdots,T_2(\alpha_n))\\ &= T_1((\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_2))\\ &= (T_1(\alpha_1,\alpha_2,\cdots,\alpha_n))\varphi(T_2)\\ &= ((\alpha_1,\alpha_2,\cdots,\alpha_n)\varphi(T_1))\varphi(T_2))\\ &= (\alpha_1,\alpha_2,\cdots,\alpha_n)(\varphi(T_1))\varphi(T_2))\end{aligned}$$

所以$\varphi(T_1T_2)=\varphi(T_1)\varphi(T_2)$

(4)验证$\varphi(\mathcal{E})=E$

$$\mathcal{E}(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\mathcal{E}(\alpha_1),\mathcal{E}(\alpha_2),\cdots,\mathcal{E}(\alpha_n))=(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)E$$

所以$\varphi(\mathcal{E})=E$

(5)当T_1是可逆线性变换时, 验证$\varphi(T_1^{-1})=\varphi(T_1)^{-1}$

$$E=\varphi(\mathcal{E})=\varphi(T_1T_1^{-1})=\varphi(T_1)\varphi(T_1^{-1})$$

所以$\varphi(T_1^{-1})=\varphi(T_1)^{-1}$

定理. 设线性变换T 在一个有序基$(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 下的矩阵是A, 向量α 在这个基下的坐标是$X=(x_1,x_2,\cdots,x_n)^t$. 则向量$T(\alpha)$ 在基$\alpha_1,\alpha_2,\cdots,\alpha_n$ 下的坐标$Y=(y_1,y_2,\cdots,y_n)^t$ 为

$$Y=AX$$

证明. 由于向量α 和$T(\alpha)$在基$(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 下的坐标分别是X和Y, 所以

$$\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)X \quad T(\alpha)=(\alpha_1,\alpha_2,\cdots,\alpha_n)Y$$

于是

$$\begin{aligned}T(\alpha)&=T((\alpha_1,\alpha_2,\cdots,\alpha_n)X)\\&=(T(\alpha_1,\alpha_2,\cdots,\alpha_n))X\\&=(\alpha_1,\alpha_2,\cdots,\alpha_n)AX\end{aligned}$$

所以

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)Y=(\alpha_1,\alpha_2,\cdots,\alpha_n)AX$$

由于$(\alpha_1,\alpha_2,\cdots,\alpha_n)$线性无关, 所以$Y=AX$.

说明. 定理给出线性变换作用的方式.

例. 设$V=\mathbb{F}_3[x]$, 已知V上的线性变换T在基$(1,x,x^2,x^3)$下的矩阵为:

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$$

对于$f(x)\in V$, 求$T(f(x))$

解. 设$f(x)=a_3x^3+a_2x^2+a_1x+a_0$, 则$f(x)$在基$(1,x,x^2,x^3)$下的坐标为$(a_0,a_1,a_2,a_3)^t$, 所以$T(f(x))$在基$(1,x,x^2,x^3)$下的坐标为

$$A\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\\a_0\end{pmatrix}$$

所以$T(f(x))=a_0x^3+a_3x^2+a_2x+a_1$

对于线性映射$T:V\longrightarrow W$, 也有类似于线性变换的结论, 下面我们给出这些结论, 它们的证明与线性变换情况的证明类似, 请同学们完成.

定理. 设V和W分别是域$\mathbb{F}$上的n维与m线性空间, $(\alpha_1,\alpha_2,\cdots,\alpha_n)$与$(\beta_1,\beta_2,\cdots,\beta_m)$分别是$V$和$W$的有序基. 则

(1) 存在一个双射$\psi:\mathcal{L}(V,W)\to M_{m\times n}(\mathbb{F})$使得

$$T(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\beta_1,\beta_2,\cdots,\beta_m)\psi(T),\quad T\in\mathcal{L}(V,W)$$

(2)ψ是线性空间$\mathcal{L}(V,W)$到线性空间$M_{m\times n}(\mathbb{F})$的同构.

(3) 对任意$\alpha\in V$, 如果α在基$(\alpha_1,\alpha_2,\cdots,\alpha_n)$下的坐标是$X=(x_1,x_2,\cdots,x_n)^t$, $T(\alpha)$在基$(\beta_1,\beta_2,\cdots,\beta_m)$的坐标是$Y=(y_1,y_2,\cdots,y_m)^t$. 则$Y=\psi(T)X$

练习1. 请完成上面定理的证明.

练习2. 习题7.2: 7,8.

定义. 设A, B 是域$\mathbb{F}$ 上n 阶方阵, 如果存在n 阶可逆阵P 使得$B = P^{-1}AP$, 则称为A 相似于B, 记为$A \sim B$.

相似矩阵的一些简单性质

(1) 矩阵的相似关系是一个等价关系

证明. (自反性)由于$A = E^{-1}AE$, 所以$A \sim A$.

(对称性) 由$A \sim B$推出存在可逆阵P, 使得$B = P^{-1}AP$,令$Q = P^{-1}$, 则$A = Q^{-1}BQ$, 所以$B \sim A$.

(传递律) 如果$A \sim B, B \sim C$, 则存在可逆阵P, Q, 使$B = P^{-1}AP, C = Q^{-1}BQ$. 于是$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$, 所以$A \sim C$.

(2) 如果$A \sim B$, 则$|A| = |B|$.

证明. 由$A \sim B$推出存在可逆阵P, 使得$B = P^{-1}AP$, 等式两边取行列式得$|B| = |P^{-1}AP| = |P^{-1}||A||P| = |P|^{-1}|A||P| = |A|$.

注意. 如果$A \sim B$, 则$A \leftrightarrow B$. 反之不成立, 请同学们给出一个反例.(参考例7.2.16)

(3) 如果$A \sim B$而且A可逆, 则B可逆而且$A^{-1} \sim B^{-1}$

证明. 由于$A \sim B$, 所以$|B| = |A| \neq 0$, 由此推出B可逆. 再由$A \sim B$推出存在可逆阵P使得$B = P^{-1}AP$, 于是

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$$

(4) 如果$A \sim B$, 则对于任意非负整数m, $A^m \sim B^m$

证明.由于$A \sim B$, 所以存在可逆阵P使得$B = P^{-1}AP$, 于是

$$\begin{aligned} B^m &= (P^{-1}AP)^m = \underbrace{P^{-1}APP^{-1}APP^{-1}AP \cdots P^{-1}AP}_{m} \\ &= P^{-1}A(PP^{-1})A(PP^{-1})A(PP^{-1}) \cdots (PP^{-1})AP = P^{-1}\underbrace{AAA \cdots A}_{m}P = P^{-1}A^mP \end{aligned}$$

(5) 如果$A \sim B$, 则对于任意$f(x) \in \mathbb{F}[x]$有$f(A) \sim f(B)$.

证明. 设$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$, 由于$A \sim B$, 所以存在可逆阵$P$使得$B = P^{-1}AP$, 并且对于$k = 0, 1, 2, \cdots, m$有$B^k = P^{-1}A^kP$. 于是

$$\begin{aligned} f(B) &= a_m B^m + a_{m-1}B^{m-1} + \cdots + a_1 B + a_0 E \\ &= a_m(P^{-1}A^mP) + a_{m-1}(P^{-1}A^{m-1}P) + \cdots + a_1 P^{-1}AP + a_0(P^{-1}EP) \\ &= P^{-1}(a_m A^m + a_{m-1}A^{m-1} + \cdots + a_1 A + a_0 E)P = P^{-1}(f(A))P \end{aligned}$$

定理. 两个n 阶方阵A, B 相似的充分必要条件它们是n 维线性空间V 上某一个线性变换T 在不同基下的矩阵

证明. (必要性) 由于$A \sim B$, 所以存在可逆阵P, 使得$B = P^{-1}AP$. 设$(\alpha_1, \alpha_2, \cdots, \alpha_n)$是$V$ 一个有序基, 则对于矩阵A, 由于$\mathcal{L}(V, V)$与$M_n(\mathbb{F})$同构, 所以存在V 上的唯一线性变换T, 使得

$$T(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)A.$$

令

$$(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)P$$

由于P 可逆, 所以$\beta_1, \beta_2, \cdots, \beta_n$ 也是V 的一个基, 而且

$$\begin{aligned}
T(\beta_1, \beta_2, \cdots, \beta_n) &= T((\alpha_1, \alpha_2, \cdots, \alpha_n)P) \\
&= (T(\alpha_1, \alpha_2, \cdots, \alpha_n))P = (\alpha_1, \alpha_2, \cdots, \alpha_n)AP \\
&= (\beta_1, \beta_2, \cdots, \beta_n)P^{-1}AP
\end{aligned}$$

即T 在基$(\beta_1, \beta_2, \cdots, \beta_n)$ 下的矩阵是$P^{-1}AP = B$

(充分性)设A和B分别是n 维线性空间V 上某一个线性变换T在基$(\alpha_1, \alpha_2, \cdots, \alpha_n)$和基$(\beta_1, \beta_2, \cdots, \beta_n)$下的矩阵, 即

$$T(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)A, \quad T(\beta_1, \beta_2, \cdots, \beta_n) = (\beta_1, \beta_2, \cdots, \beta_n)B$$

设基$(\alpha_1, \alpha_2, \cdots, \alpha_n)$到基$(\beta_1, \beta_2, \cdots, \beta_n)$的过渡矩阵为$P$, 即

$$(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)P$$

于是

$$\begin{aligned}(\beta_1, \beta_2, \cdots, \beta_n)B &= T(\beta_1, \beta_2, \cdots, \beta_n)\\ &= T((\alpha_1, \alpha_2, \cdots, \alpha_n)P)\\ &= (T(\alpha_1, \alpha_2, \cdots, \alpha_n))P\\ &= (\alpha_1, \alpha_2, \cdots, \alpha_n)AP\\ &(\beta_1, \beta_2, \cdots, \beta_n)P^{-1}AP\end{aligned}$$

由于$(\beta_1, \beta_2, \cdots, \beta_n)$线性无关, 所以$B = P^{-1}AP$.

因为相似阵有相同的行列式, 所以我们可以定义线性变换T 的行列式

定义. 设T为线性空间V上的线性变换, T行列式记为$|T|$, 定义为在V 的一个有序基下的矩阵A 的行列式$|A|$.

例. 设

$$(\mathbf{e}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\mathbf{e_3}=\begin{pmatrix}0\\0\\1\end{pmatrix})\text{和}(\alpha_{\mathbf{1}}=\begin{pmatrix}1\\0\\-1\end{pmatrix}\alpha_{\mathbf{2}}=\begin{pmatrix}-1\\1\\2\end{pmatrix}\alpha_{\mathbf{3}}=\begin{pmatrix}1\\1\\1\end{pmatrix})$$

是线性空间$\mathbb{R}^3$ 的两个有序基, T是线性空间$\mathbb{R}^3$ 上的线性变换, 它在有序基$(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$下的矩阵为

$$A=\begin{pmatrix}2&-3&2\\-1&4&-2\\-2&6&-3\end{pmatrix}$$

求T 在有序基$(\alpha_1,\alpha_2,\alpha_3)$ 下的矩阵B.

解. 首先求基$(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$到基$(\alpha_1,\alpha_2,\alpha_3)$过渡矩阵$P$, 即

$$(\alpha_1,\alpha_2,\alpha_3)=(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)P.$$

得

$$P=\begin{pmatrix}1&-1&1\\0&1&1\\-1&2&1\end{pmatrix},\quad P^{-1}=\begin{pmatrix}-1&3&-2\\-1&2&-1\\1&-1&1\end{pmatrix}.$$

于是

$$B=P^{-1}AP=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$$

例. 设

$$A = \begin{pmatrix} 2 & -3 & 2 \\ -1 & 4 & -2 \\ -2 & 6 & -3 \end{pmatrix}$$

求A^n

解. 根据上例得$A = PBP^{-1}$, 于是$A^n = PB^nP^{-1}$, 而

$$B^n = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

所以

$$\begin{aligned} A^n = PB^nP^{-1} &= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -2 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+n & -3n & 2n \\ -n & 1+3n & -2n \\ -2n & 6n & 1-4n \end{pmatrix} \end{aligned}$$

练习. 习题7.2: 4,9,10,11.